$\boxed{①\ (a)}$ The characteristic equation of $4y'' + y' = 0$ is $4r^2 + r = 0$ which has roots

$$r = \frac{-1 \pm \sqrt{1^2 - 4(4)(0)}}{2(4)} = \frac{-1 \pm \sqrt{1}}{8} = \frac{-1}{8} \pm \frac{1}{8}$$

$$= -\frac{1}{8} + \frac{1}{8},\ -\frac{1}{8} - \frac{1}{8}$$

$$= 0, -\frac{1}{4}$$

Thus, the general solution to $4y'' + y' = 0$ is

$$y = c_1 e^{0x} + c_2 e^{-\frac{1}{4}x}$$

which is

$$\boxed{y = c_1 + c_2 e^{-\frac{1}{4}x}.}$$

where c_1, c_2 can be any constants.

1 (b) The characteristic equation of

$y'' - y' - 6y = 0$ is $r^2 - r - 6 = 0$

has roots

$$r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-6)}}{2(1)} = \frac{1 \pm \sqrt{25}}{2}$$

$$= \frac{1 \pm 5}{2} = \frac{1+5}{2}, \frac{1-5}{2} = \frac{6}{2}, \frac{-4}{2} = 3, -2$$

Thus, the general solution to $y'' - y' - 6y = 0$ is

$$\boxed{y = c_1 e^{3x} + c_2 e^{-2x}}$$

where c_1, c_2 can be any constants.

$\boxed{①\,(c)}$ The characteristic equation of

$y'' + 9y = 0$ is $r^2 + 9 = 0$ which

has roots

$$r = \frac{-0 \pm \sqrt{0^2 - 4(1)(9)}}{2(1)} = \frac{\pm\sqrt{-36}}{2} = \frac{\pm 6\sqrt{-1}}{2}$$

$$= \pm 3i = 3i, -3i$$

These complex roots are of the form

$$\alpha \pm \beta i = 0 \pm 3i$$

Thus, the general solution to $y'' + 9y = 0$ is

$$y = c_1 e^{0x} \cos(3x) + c_2 e^{0x} \sin(3x)$$

which is

$$\boxed{y = c_1 \cos(3x) + c_2 \sin(3x)}$$

Where c_1, c_2 are any constants.

1 (d) The characteristic equation of $y''-2y'+2y=0$ is $r^2-2r+2=0$ which has roots

$$r=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(2)}}{2(1)}=\frac{2\pm\sqrt{-4}}{2}=\frac{2\pm2\sqrt{-1}}{2}$$

$$=\frac{2+2i}{2},\frac{2-2i}{2}=1+i,1-i$$

These complex roots are of the form

$$\alpha\pm\beta i=1\pm1\cdot i$$

Thus, the general solution to $y''+9y=0$ is

$$y=c_1e^{1\cdot x}\cos(1\cdot x)+c_2e^{1\cdot x}\sin(1\cdot x)$$

which is

$$\boxed{y=c_1e^{x}\cdot\cos(x)+c_2e^{x}\cdot\sin(x)}$$

Where c_1,c_2 are any constants.

$\boxed{①\,(e)}$ The characteristic equation of

$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0$ is $r^2 + 8r + 16 = 0$ which

has roots

$$r = \frac{-8 \pm \sqrt{8^2 - 4(1)(16)}}{2(1)} = \frac{-8 \pm 0}{2} = -4$$

Here we have $r = -4$ as a double root.

Thus, the general solution to

$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0$ is given by

$$\boxed{y = c_1 e^{-4x} + c_2 x e^{-4x}}$$

where c_1, c_2 are constants

$\boxed{1 \text{ (f)}}$ The characteristic equation of $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 0$ is $r^2 - 10r + 25 = 0$ which has roots

$$r = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(25)}}{2(1)} = \frac{10 \pm 0}{2} = 5$$

Here we have $r = 5$ as a double root.

Thus, the general solution to

$\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 0$ is given by

$$\boxed{y = c_1 e^{5x} + c_2 x e^{5x}}$$

where c_1, c_2 are constants

1 (g) The characteristic equation of $2y''+2y'+y=0$ is $2r^2+2r+1=0$ which has roots

$$r = \frac{-2 \pm \sqrt{2^2-4(2)(1)}}{2(2)} = \frac{-2 \pm \sqrt{-4}}{4} = \frac{-2 \pm \sqrt{4}\sqrt{-1}}{4}$$

$$= \frac{-2 \pm 2i}{4} = \frac{-2+2i}{4}, \frac{-2-2i}{4} = -\frac{1}{2}+\frac{1}{2}i, -\frac{1}{2}-\frac{1}{2}i$$

These complex roots are of the form

$$\alpha \pm \beta i = -\frac{1}{2} \pm \frac{1}{2} i$$

Thus, the general solution to $y''+9y=0$ is

$$y = c_1 e^{-\frac{1}{2}x} \cos\left(\frac{1}{2}x\right) + c_2 e^{-\frac{1}{2}x} \sin\left(\frac{1}{2}x\right)$$

which is

$$\boxed{y = c_1 e^{-x/2} \cos\left(\frac{x}{2}\right) + c_2 e^{-x/2} \sin(x/2)}$$

Where c_1, c_2 are any constants.

② (a)

In problem ① we saw that the general solution to $4y'' + y' = 0$ is given by

$$\boxed{y = c_1 + c_2 e^{-x/4}}$$

We want this solution to satisfy $y'(0)=0, y(0)=0$

We have $y' = -\frac{1}{4} c_2 e^{-x/4}$

Thus we want

$$0 = y(0) = c_1 + c_2 \underbrace{e^{-0/4}}_{e^0=1} = c_1 + c_2$$

$$0 = y'(0) = -\frac{1}{4} c_2 e^{-0/4} = -\frac{1}{4} c_2$$

That is,

$$\boxed{\begin{aligned} c_1 + c_2 &= 0 \quad ① \\ -\tfrac{1}{4} c_2 &= 0 \quad ② \end{aligned}}$$

Eqn ② gives $c_2 = 0$. Plug this into ① to get $c_1 = -c_2 = -0 = 0$. Thus, the solution we want is $y = 0 + 0 \cdot e^{-x/4} = 0$.

That is $\boxed{y=0}$ the zero function

$\boxed{2 \text{ (b)}}$ The characteristic equation of $y'' + 16y = 0$ is $r^2 + 16 = 0$. The roots are

$$r = \frac{-0 \pm \sqrt{0^2 - 4(1)(16)}}{2(1)} = \frac{\pm\sqrt{-64}}{2} = \frac{\pm\sqrt{64}\sqrt{-1}}{2}$$

$$= \pm\frac{8i}{2} = \pm 4i$$

The roots are of the form $\alpha \pm \beta i = 0 \pm 4i$

The general solution is

$$\boxed{\begin{aligned} y &= c_1 e^{0x}\cos(4x) + c_2 e^{0x}\sin(4x) \\ &= c_1\cos(4x) + c_2\sin(4x) \end{aligned}}$$

We want this solution to satisfy $y'(0) = -2$, $y(0) = 2$

Note that

$$y' = -4c_1\sin(4x) + 4c_2\cos(4x)$$

So we want

$$-2 = y'(0) = -4c_1\underbrace{\sin(0)}_{0} + 4c_2\underbrace{\cos(0)}_{1} = 4c_2 \leftarrow c_2 = \frac{-1}{2}$$

and

$$2 = y(0) = c_1\underbrace{\cos(0)}_{1} + c_2\underbrace{\sin(0)}_{0} = c_1 \leftarrow c_1 = 2$$

Thus, the answer is

$$\boxed{y = 2\cos(4x) - \tfrac{1}{2}\sin(4x)}$$

2(c) From problem 1 the general solution to $y'' - y' - 6y = 0$ is $\boxed{y = c_1 e^{3x} + c_2 e^{-2x}}$

We want $y'(0) = 10$, $y(0) = 5$.

We have $y' = 3c_1 e^{3x} - 2c_2 e^{-2x}$.

$e^0 = 1$

These equations are

$$10 = y'(0) = 3c_1 e^{3\cdot 0} - 2c_2 e^{-2\cdot 0} = 3c_1 - 2c_2$$

$$5 = y(0) = c_1 e^{3\cdot 0} + c_2 e^{-2\cdot 0} = c_1 + c_2$$

So we want to solve

$$\boxed{\begin{aligned} 3c_1 - 2c_2 &= 10 \quad (1) \\ c_1 + c_2 &= 5 \quad (2) \end{aligned}}$$

Solving for c_1 in (2) we get $c_1 = 5 - c_2$.

Plugging this into (1) gives $3(5 - c_2) - 2c_2 = 10$.

This gives $15 - 3c_2 - 2c_2 = 10$.

So, $-5c_2 = -5$.

So, $c_2 = 1$.

Thus, $c_1 = 5 - c_2 = 5 - (1) = 4$

So, the answer is

$$\boxed{y = 4e^{3x} + e^{-2x}}$$